

Sequences and Series

ISI & CMI Crash Course 2024



Sequences



Introduction

- Suppose A is countable, S is a set. $f: A \rightarrow S$.
 $\hookrightarrow$ can assume $A = \mathbb{N}$. $\nrightarrow$ if not, let ϕ be the bijection $\mathbb{N} \rightarrow A$. Consider $f \circ \phi$.

Denote $f(1) =: s_1, f(2) =: s_2, \dots$

Thus we have a sequence $s_1, s_2, \dots$ written as: $\{s_n\}_{n \in \mathbb{N}} \subseteq S \rightarrow$ "space" set. (where are the elements.)
 $\hookrightarrow$ indexing set (for us, countable)

- We will mostly assume S to be $\mathbb{R}$.



Convergence

$\hookrightarrow \{a_n\}_{n=1}^{\infty}$

- Suppose $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$. If $\exists a \in \mathbb{R} \mid$
 $\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \forall n \geq N \mid a_n - a \mid < \varepsilon$.

This basically means, you can get arbitrarily close to the limiting value (here, " a ") if you go far enough down the sequence.

(after this point, $\{a_n\}_{n \geq N} \subseteq (a - \varepsilon, a + \varepsilon)$)

... then, we say " $\{a_n\}$ converges to a ", commonly written as: $\lim_{n \rightarrow \infty} a_n = a$, $a_n \rightarrow a$ as $n \rightarrow \infty$.

" a " is called the limiting value.

$\hookrightarrow$ which is unique (HW)

⚠ DO NOT TOUCH if you have PYQs / TOMATO / Sci Astra Book remaining.

References:

- i) S. Kumaresan $\leftarrow$ easier
 - ii) Bartle-Shorbut $\leftarrow$ default ✓
 - iii) Walter Rudin $\leftarrow$ advanced.
- } all uploaded in App.

• Example: $\{a_n := \frac{1}{n}\}_{n=1}^{\infty} \subseteq \mathbb{R}$.

Ans: $a_n \rightarrow 0$ as $n \rightarrow \infty$. $\leftarrow$ details irrelevant.

Proof: Fix $\varepsilon > 0$. By Archimedean property of reals, $\exists N \in \mathbb{N} \mid N\varepsilon > 1$.
 $\Rightarrow \forall n \in \mathbb{N}, n \geq N, n\varepsilon > 1$
 $\Rightarrow \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N}, n \geq N, \frac{1}{n} < \varepsilon$.

$\therefore$ The proof follows.

$\ll |a_n - a|$ where $a=0$.

• Example: $\{a_n := r^n\}_{n=1}^{\infty}$, where $|r| < 1$.

Ans: $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof: Fix $\varepsilon > 0$. Deal w/ $r=0$ separately.

TST: $\exists N \in \mathbb{N} \mid |r|^N < \varepsilon$

The above suffices as $0 < |r|^n \leq |r|^N \forall n \geq N, n \in \mathbb{N}$
($\because |r| < 1$)

Now, $\varepsilon > 0 \Rightarrow$ you can take $\log_{|r|}$ of ε .
 $\because \exists x = \log_{|r|} \varepsilon \in \mathbb{R}$
 $0 < |r| < 1$

$$\alpha = \log_{|r|} \varepsilon \in \mathbb{R}$$

$$\Rightarrow \exists N \in \mathbb{N} \mid N > \alpha$$

$$\Rightarrow N > \alpha = \log_{|r|} \varepsilon$$

$$\Rightarrow |r|^N < |r|^{\log_{|r|} \varepsilon} = \varepsilon$$

inequality reversed
as $|r| \in (0, 1)$

- Example: $\{a_n := a\}_{n=0}^{\infty} \subseteq \mathbb{R}$

$$\text{Fix } \varepsilon > 0. \quad |a_n - a| = 0 < \varepsilon \quad \forall n \in \mathbb{N}$$

- Monotonicity

Suppose $\{a_n\} \subseteq \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n \leq a_{n+1}$
Then $\{a_n\}$ is said to be monotonically increasing.

Suppose $\{a_n\} \subseteq \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n < a_{n+1}$
Then $\{a_n\}$ is said to be strictly increasing.

(Similarly for decreasing)

A sequence that is either monotonically increasing or monotonically decreasing is called "monotone".

- Boundedness $\{a_n\} \subseteq \mathbb{R}$ is said to be ...

($\hookrightarrow$ strict $\Rightarrow$ monotone)

i) ... bounded above: iff $\exists M \in \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n < M$

ii) ... bounded below: iff $\exists m \in \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n > m$

iii) ... bounded: iff it is both bounded above & bounded below

Supremum & Infimum

→ In $\mathbb{R}$, $A \in \mathbb{R}$ which is upper bounded, we say

$\sup A = m$ iff $\forall x \geq m, \forall y \in A, x \geq y$
 $\equiv \inf A$ } $\forall x < m, \exists y \in A \mid y > x$
 $\hookrightarrow m$ is the least upper bound (lub)

→ Similarly, for inf & lower bounds
Cglb: greatest lower bound.

→ NOTE: sup, inf are defined for orderings in general, but we shall look at them in context of $\mathbb{R}$.

→ sup, inf don't always exist. Eg: $\{x \mid x^2 < 2\} \subseteq \mathbb{Q}$ is upper bounded but sup doesn't exist

($\because \sqrt{2}$ is irrational) ($\sim$ incompleteness of rationals)
 ($\sim$ completion of $\mathbb{Q}$ is $\mathbb{R}$, so $\mathbb{R}$ has sup property)
 sup, inf exist in $\mathbb{R}$. $\checkmark$ $\Leftrightarrow$ inf property

(c) If not bounded, $\sup A = +\infty$, $\inf A = -\infty$ (accordingly)

→ sup/inf are not exactly max/min, but are similar.

E_5 : $(0, 1) \rightarrow$ no max, no min
sup = 1, inf = 0.

But, $\sup A, \inf A \notin A$ necessarily... $\in A$
If $\in A$, they are $\max A, \min A$ resp. (by defn)

• Monotone Convergence Theorem

Suppose $\{a_n\} \subseteq \mathbb{R}$ is monotonically increasing & bounded above.

Then: $a_n \rightarrow a := \sup \{a_n \mid n \in \mathbb{N}\}$.

(same for decreasing, bounded below, inf)

Proof:

Fix any $\varepsilon > 0$.

$\because a = \sup \{a_n\}$, $a - \varepsilon$ is not an upper bound of $\{a_n\}$

$$\Rightarrow \exists N \in \mathbb{N} \mid a - \varepsilon < a_N \leq a$$

Now, $\because \{a_n\}$ is monotonically increasing --

$$\forall n \geq N, n \in \mathbb{N}, a - \varepsilon < a_N \leq a_n \leq a$$

$$\Rightarrow \forall n \geq N, n \in \mathbb{N}, a - a_n < \varepsilon$$

This completes the proof.

(HW) • Result: A sequence is convergent $\Rightarrow$ it is bounded.

(HW) • Lemma

$\{a_n\} \subseteq \mathbb{R}, a_n \rightarrow a; \{b_n\} \subseteq \mathbb{R}, b_n \rightarrow b$

If $a_n \leq b_n$, then $a \leq b$.

• Standard Arithmetic Results

$$\{a_n\} \in \mathbb{R}, \{b_n\} \in \mathbb{R}. \quad a_n \rightarrow a, \quad b_n \rightarrow b$$

$$i) \quad a_n + b_n \rightarrow a + b$$

$$ii) \quad \lambda a_n \rightarrow \lambda a \quad \forall \lambda \in \mathbb{R}$$

$$iii) \quad a_n b_n \rightarrow ab$$

$$iv) \quad \text{If } \forall n, \frac{a_n}{b_n} \rightarrow \frac{a}{b}.$$

Proof: (abuse of Δ inequality)

$$i) \quad |a_n + b_n - a - b| \leq |a_n - a| + |b_n - b|$$

$$\text{Fix any } \varepsilon > 0, \quad \exists N_1 \in \mathbb{N} \mid \forall n \geq N_1, |a_n - a| < \varepsilon/2 \\ \exists N_2 \in \mathbb{N} \mid \forall n \geq N_2, |b_n - b| < \varepsilon/2$$

Take $N = \max\{N_1, N_2\}$ & use ①

$$\therefore |(a_n + b_n) - (a + b)| < \varepsilon \quad \forall n \geq N$$

$$ii) \quad |\lambda a_n - \lambda a| = \lambda |a_n - a| < \varepsilon$$

$< \varepsilon/\lambda$ for large enough n

$$iii) \quad |a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \\ \leq a_n |b_n - b| + b |a_n - a| \\ \leq \underbrace{|a - \varepsilon| \varepsilon + b \varepsilon}_{\text{arbitrarily small (can be made)}} \text{ for large enough } n$$

iv)

HW

• Sandwich Theorem / Squeeze Theorem

If $a_n \leq b_n \leq c_n$, $a_n \rightarrow \lambda$, $c_n \rightarrow \lambda$,
then $b_n \rightarrow \lambda$.

Proof: If we show $\{b_n\}$ converges, then by Lemma,
 $b_n \rightarrow \lambda$.

$\forall n$, $b_n \in [a_n, c_n]$.

Fix any $\varepsilon > 0$. For large enough n : a_n, c_n
 $\in (\lambda - \varepsilon, \lambda + \varepsilon)$

Now, $[a_n, c_n] \subseteq (\lambda - \varepsilon, \lambda + \varepsilon)$

$\therefore b_n \in (\lambda - \varepsilon, \lambda + \varepsilon)$

$(\Rightarrow) |b_n - \lambda| < \varepsilon$ ■

• Examples: See S. Kumarasam and/or Beutle-Shubert.

• **Exercise Set 2.4.3.** Use the sandwich lemma to solve the following.

HW

- (1) Let (a_n) be a bounded (real) sequence and (x_n) converge to 0. Then show that $a_n x_n \rightarrow 0$.
- (2) The sequence $\sqrt{n+1} - \sqrt{n} \rightarrow 0$.
- (3) $x_n := \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \rightarrow 1$.
- (4) Let $0 < a < b$. The sequence $((a^n + b^n)^{1/n}) \rightarrow b$.

• **Theorem 2.4.4** (Nested Interval Theorem – Standard Version). Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Assume further that $b_n - a_n \rightarrow 0$, that is, the sequence of lengths of J_n 's goes to zero. Then there exists a unique c such that $\cap_n J_n = \{c\}$.

$\rightarrow$ S. Kumarasam.

Proof. We have already seen (Theorem 1.3.23) that there exists at least one $c \in \cap_n J_n$. Let $c, d \in \cap_n J_n$. Assume that $c \leq d$. Then, since $a_n \leq c \leq d \leq b_n$, we see that $0 \leq d - c \leq b_n - a_n$. Since $b_n - a_n \rightarrow 0$, it follows from the sandwich lemma that $d - c = 0$. $\square$

Corollary
of
Sandwich
Theorem.

(not imp.)

• Extended Defⁿ of Limits

We say a sequence "diverges" if it doesn't converge. However, not all divergences are the same!

$$\{a_n\} \subseteq \mathbb{R}. \quad \text{If } \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \mid$$

$$\forall n \geq N, n \in \mathbb{N}, a_n > M$$

... then we say $\{a_n\}$ diverges to $+\infty$, written as: $a_n \rightarrow +\infty$.

extended defⁿ of limits!

(similar for $a_n < m, a_n \rightarrow -\infty$)

From now on, to indicate a finite limit, we may write something like $a_n \rightarrow a \in \mathbb{R}$.
 always look @ context as well.

Conventions:

$$\begin{array}{l} \infty + a = \infty \\ -\infty + a = -\infty \end{array} \quad \parallel \quad \begin{array}{l} \infty \cdot a = \infty \\ -\infty \cdot a = -\infty \end{array} \quad \left. \vphantom{\begin{array}{l} \infty + a = \infty \\ -\infty + a = -\infty \end{array}} \right\} a \neq 0$$

$$\infty + \infty = \infty$$

$$\infty \cdot \infty = \infty$$

$$-\infty - \infty = -\infty$$

$$(-\infty)(-\infty) = \infty$$

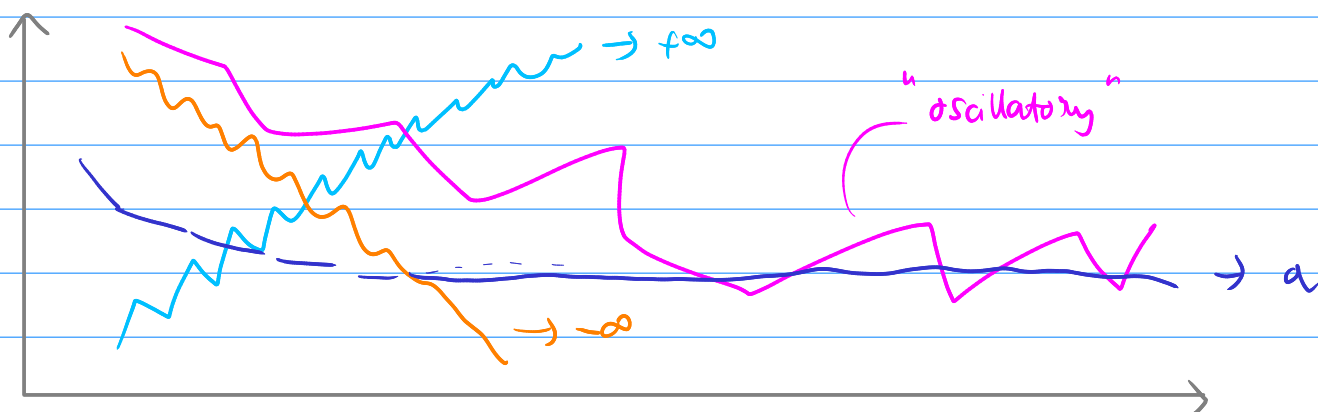
$$\infty - \infty = \text{undefined}$$

$$\infty(-\infty) = -\infty$$

But!

$\pm \infty \cdot 0$ can be 0 or $\pm \infty$ or something else!
 (same for $\pm \infty / \pm \infty$)

(for using ∞ 's in arithmetic results)



6

i) (9) True or False: If (x_n) and (y_n) are sequences such that $x_n y_n \rightarrow 0$, then one of the sequences converges to 0.

ii) (11) Let $a_n \rightarrow 0$. What can you say about the sequence (a_n^n) ?

iii) (14) Let (x_n) be a sequence. Assume that $x_n \rightarrow 0$. Let $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Define a new sequence $y_n := x_{\sigma(n)}$. Show that $y_n \rightarrow 0$.

Ans)

i) False: $x_n \rightarrow 0, 1, 0, 1, 0, 1, \dots$
 $y_n \rightarrow 1, 0, 1, 0, 1, 0, \dots$

$x_n y_n \equiv 0 \quad \therefore x_n y_n \rightarrow 0$. But $x_n \not\rightarrow 0, y_n \not\rightarrow 0$.

ii) Fix any $\varepsilon > 0$.

$\exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N} \mid |a_n| < \min\{\varepsilon, 1\}$

$\therefore |a_n| < 1, |a_n^n| < |a_n| < \min\{\varepsilon, 1\}$

$\therefore a_n^n \rightarrow 0$

iii) Fix any $\varepsilon > 0$.

$\because x_n \rightarrow 0, \exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N} \mid |x_n| < \varepsilon$.

Now, $\{\sigma^{-1}(1), \sigma^{-1}(2), \sigma^{-1}(3), \dots, \sigma^{-1}(N-1)\} \subseteq \mathbb{N}$ is finite.
 $\therefore N' := \max\{\sigma^{-1}(1), \dots, \sigma^{-1}(N-1)\} + 1$

$\therefore \forall n \geq N', n \in \mathbb{N}, \sigma(n) \geq N$

$\Rightarrow \forall n \geq N', n \in \mathbb{N}, |x_{\sigma(n)}| < \varepsilon$

$\therefore y_n := x_{\sigma(n)} \rightarrow 0$.

Crux! σ takes tails to tails...

① Subsequences

- Defⁿ: Let $\{x_n\} \subseteq \mathbb{R}$ be a sequence (in $\mathbb{R}$)
 $\& \{n_k\} \subseteq \mathbb{N}$ be a strictly increasing
sequence (in $\mathbb{N}$).

Then, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Basically x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 ...
1 2 3 4 5 6 7 8 ...
✓ ✓ ✓ ✓

Subsequence: x_2 x_4 x_6 x_8 ...

- Result: $\underbrace{x_n \rightarrow x}_P \iff \exists \{x_{n_k}\} \subseteq_{\text{seq}} \{x_n\}$
 $\underbrace{x_{n_k} \rightarrow x}_Q$

Proof:

$P \Rightarrow Q$ (easy)

$P \Rightarrow Q$

i) If $x_n \rightarrow x$, then trivial, exc.

ii) If $x_n \not\rightarrow x$, then we have to show $\exists$ atleast one subsequence $\rightarrow x$.

$\therefore x_n \not\rightarrow x, \forall \epsilon > 0$, there are infinitely many $n \mid |x_n - x| \geq \epsilon$.

Form those infinitely many n into a sequence

HW: Fill in the details.

- Monotone Subsequence Theorem (won't prove)

Every real sequence $\{x_n\} \subseteq \mathbb{R}$ has a monotone subsequence.

- Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Proof (for $\mathbb{R}$): Consider $\{x_n\} \subseteq \mathbb{R}$, bounded.

--- has a monotone subseq $\{x_{n_k}\}$ which is also bounded
 $\Rightarrow$ convergent.



- Standard Limits

Theorem 2.5.1. We have the following important limits:

(1) Let $0 \leq r < 1$ and $x_n := r^n$. Then $x_n \rightarrow 0$.

(2) Let $-1 < t < 1$. Then $t^n \rightarrow 0$.

(3) Let $|r| < 1$. Then $nr^n \rightarrow 0$.

(4) Let $a > 0$. Then $a^{1/n} \rightarrow 1$.

(5) $n^{1/n} \rightarrow 1$.

(6) Fix $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \rightarrow 0$.

(A) $\left(1 + \frac{1}{n}\right)^n \uparrow e \in \mathbb{R} \setminus \mathbb{Q}$ $e \approx 2.718281\dots$

NOTATION: $\uparrow$: converges (monotonically increasing)
 $\downarrow$: " (" decreasing)

try proving as (Hw)

• Lemma If $\{x_n\}$ is divergent it is either —

i) unbounded, or —

ii) has 2 subsequences converging to different limits.

• Theorem A bounded sequence $\{x_n\}$ is convergent iff every convergent subsequence converges to the same limit. $\rightarrow$ also the limit of $\{x_n\}$.

• Theorem A sequence $\{x_n\}$ is convergent iff every subsequence $\{x_{n_k}\}$ has a further subsequence $\{x_{n_{k_l}}\}$ s.t. $\{x_{n_{k_l}}\}$ converges to the same limit.

$$x_n \rightarrow x \Leftrightarrow \forall \{x_{n_k}\} \subseteq \{x_n\}, \exists \{x_{n_{k_l}}\} \subseteq \{x_{n_k}\}$$

$$x_{n_{k_l}} \rightarrow x \text{ as } l \rightarrow \infty$$

• Defⁿ: If a bounded seq. $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \mid x_{n_k} \rightarrow x^*$, then x^* is called a subsequential limit of $\{x_n\}$.

⑥ Limit superior & limit inferior

(Refer Bartle-Shorbut Pg 82)

• Consider $a_n := (-1)^n + \frac{2}{n}$. The subsequential limits are ± 1 .

But, the largest element is $a_2 = 2$.

The idea is to capture the interval where the sequence remains active.

- Defⁿ: $\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup \{x_m \mid m \geq n, m \in \mathbb{N}\}$
 $\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \inf \{x_m \mid m \geq n, m \in \mathbb{N}\}$

- Result: If x^* is a subsequential limit of $\{x_n\}$, then:
 $\liminf \{x_n\} \leq x^* \leq \limsup \{x_n\}$

In fact, if S is the set of all subsequential limits,
 $\liminf \{x_n\} = \inf S$; $\limsup \{x_n\} = \sup S$

- Corollary $\{x_n\}$ is convergent iff $\liminf = \limsup$
 $(= \lim)$

- While limit may or may not exist, if we allow $\pm\infty$,
 $\limsup$ & $\liminf$ of a sequence always exists.
 $\hookrightarrow$ they are immune from oscillations.

- Cauchy Criterion Remark: $\mathbb{R}$ is complete, $\mathbb{Q}$ is not.

- Defⁿ: A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is called Cauchy
 iff $\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid$
 $\forall m, n \in \mathbb{N}, m, n > N \implies |x_m - x_n| < \varepsilon$

- Basically, the terms of the sequence get bunched up
 as we go. Essentially, the sequence "wants" to converge.

- Theorem: $\because \mathbb{R}$ is complete, in $\mathbb{R}$: Cauchy $\iff$ Convergent.
 $\hookrightarrow$ - requires "complete"
 $\leftarrow$ - always true



Series

o

Introduction

sum of first n terms

• Defⁿ: Let $\{a_n\} \in \mathbb{R}$. Denote $S_n := \sum_{k=1}^n a_k$.

Then $\{S_n\}$, the sequence of partial sums is called the series of $\{a_n\}$.

• Eg.: Let $\{a_n := \frac{1}{n}\}_{n \in \mathbb{N}}$ be the sequence.

Then:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &\vdots \end{aligned}$$

• Observation $\{a_n\} \equiv \{S_n - S_{n-1}\}_{n \in \mathbb{N}}$ with
will assume in general. $\leftarrow$ convention: $S_0 = 0$.

• Observation $\{a_n\}$ non-negative $\Leftrightarrow \{S_n\}$ monotone $\uparrow$

• Defⁿ: If $S_n \rightarrow S$, we say $\sum_{k=1}^{\infty} a_k$ converges,
and $\sum_{k=1}^{\infty} a_k := S$. Also, we say $\{a_n\}$ is summable if

(We are defining infinite sums now!)

(Often, the series $\{S_n = \sum_{k=1}^n a_k\}_{n=1}^{\infty}$ is simply written as $\sum_{n=1}^{\infty} a_n$) $S \in \mathbb{R}$.

we interchange
the series w/ the
value
expression

- Lemma $\{a_n\}$ summable $\Rightarrow a_n \rightarrow 0$.

Proof: $a_n = S_n - S_{n-1}$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = S - S = 0$$

- Lemma (Series Arithmetic)

$$\begin{aligned} \{a_n\} &\leadsto \{A_n\} \text{ sequence of partial sums } \rightarrow A \\ \{b_n\} &\leadsto \{B_n\} \text{ " " " " } \rightarrow B \end{aligned}$$

i) $a_n \leq b_n \Rightarrow A_n \leq B_n \Rightarrow A \leq B$.

ii) $\{a_n + \lambda b_n\} \leadsto \{A_n + \lambda B_n\} \rightarrow A + \lambda B$

$\therefore$ Order preserving on $[-\infty, \infty]$ & linear.

- Result: $\{a_n\}$ non-negative, then
 $\{S_n\}$ bounded $\Leftrightarrow \{a_n\}$ summable.

Proof: Use MCT. (Monotone Convergence Theorem)

- Result (Cauchy Criterion for Series)

$$\{a_n\} \text{ summable} \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \mid$$

$$\forall m > n \geq N \quad \underbrace{\left| \sum_{k=n+1}^m a_k \right|}_{S_m - S_n} < \varepsilon$$

• Result: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} + \frac{1}{2k} \right) = \left(1 + \frac{1}{2} \right) + \left(\frac{1}{3} + \frac{1}{4} \right) + \left(\frac{1}{5} + \frac{1}{6} \right) + \dots$$

$$\geq \sum_{k=2}^{\infty} \left(\frac{1}{2k} + \frac{1}{2k} \right) = \left(1 + \frac{1}{2} \right) + \left(\frac{1}{4} + \frac{1}{4} \right) + \left(\frac{1}{6} + \frac{1}{6} \right) + \dots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2} \right) + \left(\frac{1}{3} \right) + \dots$$

$$= \frac{1}{2} + \left(1 + \frac{1}{2} + \frac{1}{3} + \dots \right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n}$$

This is impossible if $\sum_{n=1}^{\infty} \frac{1}{n} \in \mathbb{R} \Rightarrow$ Harmonic series diverges. $\blacksquare$

• Result: $|r| < 1 \Leftrightarrow \sum_{n=1}^{\infty} r^n$ converges.

Proof: Take the sequence of partial sums:

$$S_n := \sum_{k=1}^n r^k = \frac{r}{1-r} (1 - r^n) \quad \forall |r| \neq 1.$$

$$\text{If } |r| < 1, S_n \rightarrow \frac{r}{1-r} \quad (\because r^n \rightarrow 0)$$

$$|r| > 1, S_n \text{ diverges} \quad (\because r^n \text{ diverges})$$

If $|r| = 1$, either series is $1 + 1 + 1 + \dots \rightarrow \infty$ will do later.
 $1 - 1 + 1 - 1 \dots \rightarrow$ Cesàro convergent, but not convergent

We will work w/ non-negative sequences. We shall soon see why alternating series are problematic...

② Tests for Convergence

• Comparison Test

Consider $0 \leq \{a_n\} \leq \{b_n\}$, then:

$$\{b_n\} \text{ summable} \Rightarrow \{a_n\} \text{ summable}$$

Proof: $\{b_n\}$ summable

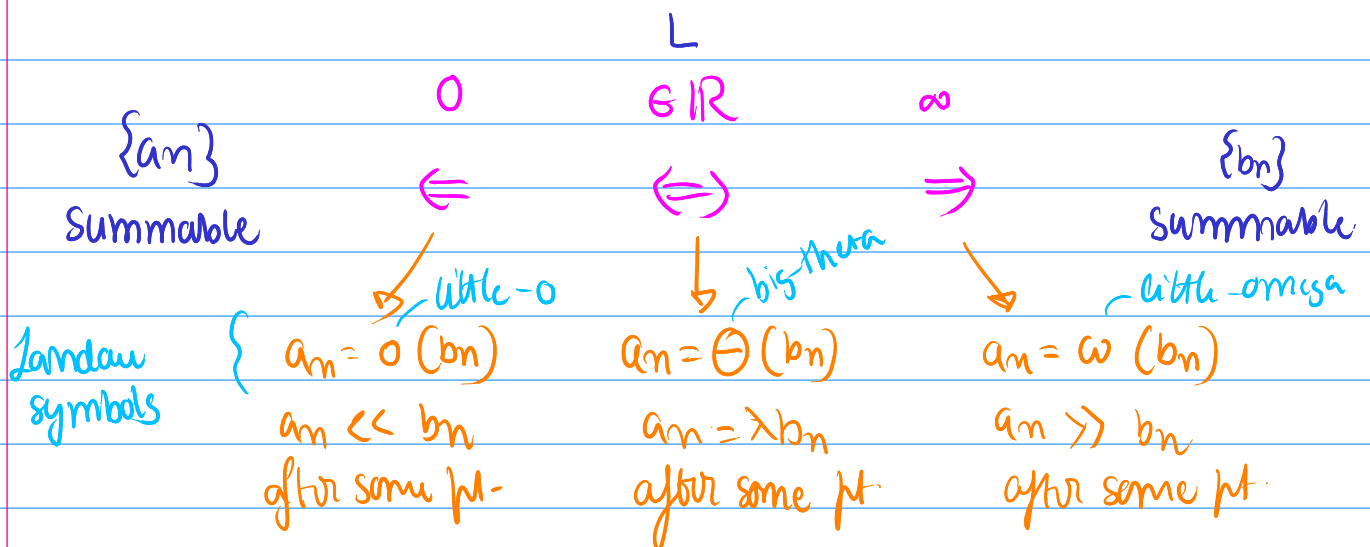
$$\Leftrightarrow \{b_n\} \text{ convergent} \Rightarrow \{b_n\} \text{ bounded.}$$

$\therefore 0 \leq \{a_n\} \leq \{b_n\}$, then $\{a_n\}$ bounded, monotone $\uparrow$.

Hence, $\{a_n\}$ convergent $\Rightarrow \{a_n\}$ summable.

• Limit Comparison Test (LCT) $\rightarrow$ growth rates comparison.

Consider $0 < \{a_n\}, \{b_n\}$. Denote $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$



• Cauchy Condensation Test.

Given $\{a_n\} \geq 0$ monotonically decreasing.

$$\sum_{n \in \mathbb{N}} a_n \text{ convergent} \Leftrightarrow \sum_{n \in \mathbb{N}} 2^n a_{2^n} \text{ convergent}$$

Estimate [edit]

The Cauchy condensation test follows from the stronger estimate,

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2 \sum_{n=1}^{\infty} f(n),$$

which should be understood as an [inequality of extended real numbers](#). The essential thrust of a [proof](#) follows, patterned after [Oresme's](#) proof of the [divergence of the harmonic series](#).

To see the first inequality, the terms of the original series are rebracketed into runs whose lengths are [powers of two](#), and then each run is bounded above by replacing each term by the largest term in that run. That term is always the first one, since by assumption the terms are non-increasing.

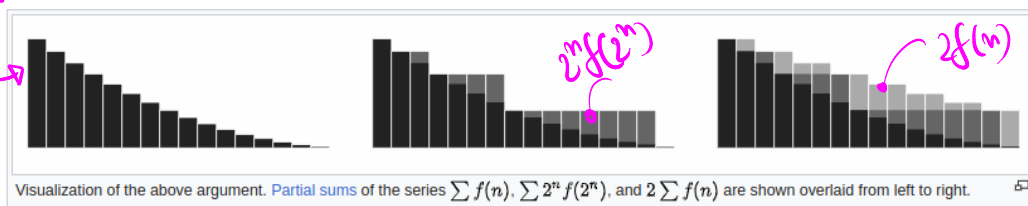
$$\begin{aligned} \sum_{n=1}^{\infty} f(n) &= f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + \dots \\ &= f(1) + (f(2) + f(3)) + (f(4) + f(5) + f(6) + f(7)) + \dots \\ &\leq f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) + \dots \\ &= f(1) + 2f(2) + 4f(4) + \dots = \sum_{n=0}^{\infty} 2^n f(2^n) \end{aligned}$$

To see the second inequality, these two series are again rebracketed into runs of power of two length, but "offset" as shown below, so that the run of $2 \sum_{n=1}^{\infty} f(n)$ which begins with $f(2^n)$ lines up with the end of the run of $\sum_{n=0}^{\infty} 2^n f(2^n)$ which ends with $f(2^n)$, so that the former stays always "ahead" of the latter.

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n f(2^n) &= f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) + \dots \\ &= (f(1) + f(2)) + (f(2) + f(4) + f(4) + f(4)) + \dots \\ &\leq (f(1) + f(1)) + (f(2) + f(2) + f(3) + f(3)) + \dots = 2 \sum_{n=1}^{\infty} f(n) \end{aligned}$$

$f(n) =: a_n$

$f(n) = a_n$



https://en.wikipedia.org/wiki/Cauchy_condensation_test

- Eg.: Prove that the harmonic series diverges using Cauchy condensation test.
(HW)

Q Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Ans: Let $a_n = \frac{1}{n^2}$. TST! $\sum_{n \in \mathbb{N}} a_n$ converges.

By Cauchy condensation, suffices to show $\sum_{n \in \mathbb{N}} 2^n a_{2^n}$ convs.

Now, $2^n a_{2^n} = 2^n (2^n)^{-2} = 2^{-n}$

$$\Rightarrow \sum_{n \in \mathbb{N}} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^{-n} = 1 \quad (\text{infinite GP})$$

$\therefore \sum \frac{1}{n^2}$ converges. (but to what? $\leftarrow$ Basel Problem)
(solved by Euler, ans $\Rightarrow \pi^2/6$)

Q (p-Series problem)

Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

by
Cauchy
Condensation

$$\Leftrightarrow \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges.}$$

$$\Leftrightarrow \sum_{n=1}^{\infty} (2^{1-p})^n \text{ converges.}$$

$$\Leftrightarrow 2^{1-p} < 1 \quad \text{by infinite GP}$$

$$\Leftrightarrow p > 1 \quad \blacksquare$$

intuition: checks for "geometric-like" decay of terms.

• Cauchy Ratio Test

→ Consider $\{a_n\}$. $L := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

If $L > 1$, $\sum a_n$ diverges.

$L < 1$, $\sum a_n$ converges absolutely.

$L = 1$, can't say

we will see later.

↳ ex: $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^2}$ converges

→ Stronger version: $R := \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Leaves, $\therefore$ lim may not exist!

$r := \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

If $r > 1$, diverges.

$R < 1$, converges absolutely.

otherwise, inconclusive.

(Proof is LCT w/ geometric series)

→ Defn: A sequence $\{a_n\}$ is called contractive iff $\exists C \in (0, 1)$
& $\forall n \in \mathbb{N} \mid \forall n \in \mathbb{N}, n \geq N$

$$|a_{n+1} - a_n| \leq C |a_n - a_{n-1}|$$

↳ contraction factor.

→ Result: Contractive $\Rightarrow$ Cauchy $\stackrel{(R)}{\Rightarrow}$ convergent.

→ Cauchy Ratio Test: $\{S_n\}$ contractive in $\mathbb{R} \Rightarrow \{S_n\}$ convergent
 $\Leftrightarrow \{a_n\}$ summable.

• Cauchy Root Test

Consider $\{a_n\} \subseteq \mathbb{R}$. $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$

If $L > 1$, diverges.

$L < 1$, converges absolutely,

$L = 1^+$ ($L = 1$, limit is from above) diverges,

otherwise inconclusive. $\leadsto \sum \frac{1}{n}, \sum \frac{1}{n^2} \dots$

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than n th roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \rightarrow \infty$.

⑥ Prove that $\sum_{n \in \mathbb{N}} \frac{1}{n^2 - n + 1}$ converges.

DIY

Hint: Use LCT with $\frac{1}{n^2} \dots$

$$\frac{\frac{1}{n^2 - n + 1}}{\frac{1}{n^2}} = \frac{1}{1 - (\frac{1}{n}) + (\frac{1}{n^2})} \rightarrow 1$$

Effectively end of syllabus. Up next are "extra" topics.

✓ Conditional Convergence & Riemann Rearrangement.

✓ Cesàro Convergence

✓ Asymptotic Notation / Landau symbols.

Conditional Summability & Riemann Rearrangement Theorem

- Def'n: $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent iff $\sum_{k=1}^{\infty} |a_k|$ is convergent.
 $\Leftrightarrow \{ |a_k| \}$ is summable. not commonly used.

We may call $\{a_n\}$ "absolutely summable".

If $\sum_{k=1}^{\infty} a_k$ is convergent but not absolutely convergent, then we say $\sum_{k=1}^{\infty} a_k$ is conditionally convergent.
 $\hookrightarrow$ call $\{a_n\}$ "conditionally summable".

If $\{a_n\}$ non-negative, summable $\Leftrightarrow$ absolutely summable.

Examples:

i) $\sum_{j=1}^{\infty} 2^{-j} = 1$ (absolutely convergent)

ii) $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (divergent)

In fact, $\{H_n\}$ be sequence of partial sums, then $H_n \sim \log(n)$

iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$ (conditionally convergent)
comes from Maclaurin series of $\ln(1+x)$

- Riemann Rearrangement Theorem

If $\sum a_n$ is conditionally convergent, then:

$$\forall a \in \mathbb{R}, \quad \exists \sigma: \mathbb{N} \rightarrow \mathbb{N} \text{ bijection} \mid \sum a_{\sigma(n)} = a$$

Essentially, you can permute the elements to get any limit!
(order of summation matters!)

Motivation

What is special about a conditionally convergent series?

$\{a_n\} \leftarrow$ conditionally summable seqⁿ.

$\hookrightarrow$ has both positive & negative terms! ^{include 0}

Take a subsequence $\begin{matrix} \{a_n^+\} & \text{along all } +\text{ve terms} \\ \{a_n^-\} & \text{" " " -ve terms} \end{matrix}$

Observe: $\sum a_n^+ = \infty, \quad \sum a_n^- = -\infty$

If not, we would have absolute convergence (both finite)
or divergence (only one finite)

So, we kind of have an $\infty - \infty$ situation.

We can go up or down an arbitrary amount simply by adding appropriate terms to the running sum, and we can do so infinitely often. (think of a submarine w/ infinite ballast tanks)

OUT OF SYLLABUS.

○ Césaro Convergence

- Defⁿ: We say $\{a_n\} \subseteq \mathbb{R}$ is Césaro convergent to a , usually written as: " $a_n \rightarrow a$ Césaro" iff:

$$\lim_{n \rightarrow \infty} \underbrace{\frac{1}{n} \sum_{k=1}^n a_k}_{=: \alpha_n} = a \in \mathbb{R}$$

... i.e. the sequence of partial averages converges to a .

- Theorem: $a_n \rightarrow a$ ($\in \mathbb{R}$) $\Rightarrow a_n \rightarrow a$ Césaro.

Proof: Fix $\varepsilon > 0$. TST: $\alpha_n \rightarrow a$.

$\because a_n \rightarrow a$, $\exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N} \mid |a_n - a| < \frac{\varepsilon}{2}$.
 $\because a_n$ convergent, a_n is bounded. Say, $\forall n \in \mathbb{N}, |a_n - a| < M$

Now, by Archimedean property, $\exists N' \in \mathbb{N} \mid \forall n \geq N', n \in \mathbb{N} \mid \frac{MN}{n} < \frac{\varepsilon}{2}$.

Take $n \geq \max\{N, N'\}$, $n \in \mathbb{N}$.

$$\begin{aligned} \therefore |\alpha_n - a| &= \left| \frac{1}{n} \sum_{k=1}^n a_k - a \right| = \frac{1}{n} \left| \sum_{k=1}^n a_k - na \right| \\ &= \frac{1}{n} |a_1 + \dots + a_N + a_{N+1} + \dots + a_n - na| \\ &= \frac{1}{n} | \underbrace{(a_1 - a) + \dots + (a_N - a)}_{\text{first } N \text{ terms}} + \underbrace{(a_{N+1} - a) + \dots + (a_n - a)}_{\text{last } n-N \text{ terms}} | \\ &\leq \frac{1}{n} \left[\{ |a_1 - a| + \dots + |a_N - a| \} + \{ \dots + |a_n - a| \} \right] \\ &\leq \frac{1}{n} \left[NM + (n-N) \frac{\varepsilon}{2} \right] < \varepsilon \quad \square \end{aligned}$$



NOTE: Cesàro convergence $\nRightarrow$ convergence.

$\xi: 1, 0, 1, 0, 1, 0, \dots$ clearly, diverges.

But, $\alpha_n = \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil$, $n \in \mathbb{N}$. $\alpha_n \rightarrow \frac{1}{2}$.
 $\therefore a_n \rightarrow \frac{1}{2}$ Cesàro

- Cesàro summability is defined in the same vein as summability. i.e. iff the sequence of partial sums is Cesàro convergent

$\{a_n\}: 1, -1, 1, -1, 1, -1, 1, -1, \dots$ ← sequence $\uparrow$
 $1, 0, 1, 0, 1, 0, 1, 0, \dots$ ← series $\uparrow$
 $\rightarrow \frac{1}{2}$ Cesàro. $\rightarrow 1 + (-1) + 1 + (-1) + \dots = \frac{1}{2}$ in Cesàro.

- Let us consider a permutation of the sequence--
 $(+1, +1, -1, +1, +1, -1, +1, +1, -1, \dots)$

$\uparrow$
 b_1 ← sequence of partial sums.

$$\therefore B_n = n - 2 \left\lfloor \frac{n}{3} \right\rfloor$$

$$\therefore B_n = \frac{B_1 + \dots + B_n}{n} \approx \frac{\frac{n(n+1)}{2} - 2 \cdot 3 \cdot \frac{\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right)}{2}}{n}$$

$\rightarrow \infty$ as $n \rightarrow \infty$

$\therefore$ Cesàro summations change upon permutations.

- Consider the sequence:
 $(+1, -2, +3, -4, +5, -6, \dots)$

$$c_n = (-1)^{n-1} n$$

$$C_n = (-1)^{n-1} \left\lfloor \frac{n+1}{2} \right\rfloor \rightarrow (1, -1, 2, -3, 3, -3, \dots)$$

$$G_n = \frac{c_1 + \dots + c_n}{n} = \begin{cases} 0 & n \text{ is even} \\ \frac{1}{n} \left\lfloor \frac{n+1}{2} \right\rfloor & n \text{ is odd} \end{cases}$$

Observe G_n doesn't converge. But we are already in Césaro-land!

$$C_n = \frac{G_1 + \dots + G_n}{n} \rightarrow \left(\frac{1}{4} \right)$$

$$S_2 = 1 - 2 + 3 - 4 + 5 - 6 + \dots$$

$$S_2 = 1 - 2 + 3 - 4 + 5 - \dots$$

$$2S_2 = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots = 1/2 \text{ in Césaro}$$

$$\Rightarrow S_2 = \forall n \text{ } \boxed{?} \text{ can't be a coincidence}$$

- $S = 1 + 2 + 3 + 4 + 5 + \dots \rightarrow$ doesn't converge
 $-S_2 = -1 + 2 - 3 + 4 - 5 + \dots$ at all! (∞)

$$S - S_2 = 4S$$

$$\Rightarrow S = -\frac{1}{3} S_2 = -\frac{1}{12}$$

$$\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$$

$$\nearrow \text{RG } \textcircled{1} \text{ } \text{Re}(s) > 1$$

Turns out $-\frac{1}{12}$ is the value of the analytic continuation of the Riemann zeta $f^n @ -1 \dots$

Moral: Don't mess w/ Césaro.

OUT OF SYLLABUS

① Asymptotic Notations / Lambda Symbols

• $f(n) = O(g(n))$ big-o ↪ not functions! ↗ after some n , $f(n) \leq$ multiple of $g(n)$

$\Leftrightarrow \exists c \in \mathbb{R}^+, N \in \mathbb{N} \mid \forall n \geq N, f(n) \leq cg(n)$

Since we can make c arbitrarily large, $N=1$ works.

• $f(n) = \Omega(g(n))$ big-omega $\Leftrightarrow g(n) = O(f(n))$

$\Leftrightarrow \exists c \in \mathbb{R}^+, N \in \mathbb{N} \mid \forall n \geq N, f(n) \geq cg(n)$

↪ ... lower bound multiple of $g(n)$

• $f(n) = \Theta(g(n))$ big-theta $\Leftrightarrow f(n) = O(g(n)) \& f(n) = \Omega(g(n))$

$f(n)$ is eventually bounded between two $\&$

multiples of $g(n)$. || Observe: $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

• Result: $f(n) \sim O(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty$

Corollary: $f(n) = \Theta(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \in (0, \infty)$

• $f(n) = o(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ little-o (implies big-o)

$f(n) = \omega(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$ little-omega (implies Ω)

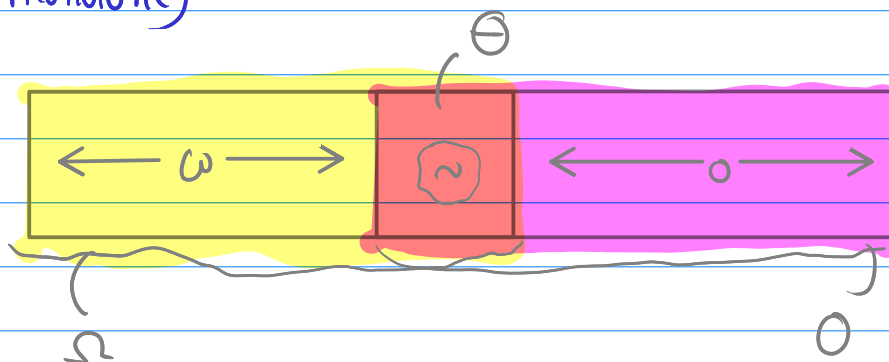
$f(n) \sim g(n) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$ asymptotically equal to. (implies Θ)

O, Ω inverses; o, ω inverses.

• Examples:

			<u>Strictest</u>
i)	$n^2 = O(n^3)$	} TRUE	O
ii)	$n^2 = O(2n^2 + n)$		Θ
iii)	$n^2 = O(n^2)$		$\sim (=)$
iv)	$n^2 = O(0.1n^2 + 1)$		Θ

- Based on this, can classify any 2 pairs of "nice" functions.
(say monotone)



Here,

$$\begin{aligned} \sim &\subseteq \Theta, & \Theta &= O \cap \Omega \\ O &= o \cup \Theta \\ \Omega &= \omega \cup \Theta \end{aligned}$$

- ⑥
- i) $n^2 + 2n + 3 + \frac{1}{n} = \Theta(n^2)$ TRUE
 - ii) $\frac{1}{n \log n} = \Omega(\frac{1}{n})$ FALSE
 - iii) $\log(\log(n!)) = O(2^n)$ TRUE
 - iv) $n^n = 2^{n \log n} = O(2^n)$ FALSE
 - v) $n^n = O(2^{2^n})$ TRUE
 - vi) $n^n = \Omega(2^{2^n})$ FALSE

Hint: $n! \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{n}{e}\right)^n$ Stirling's Approximation